Mathematical Connections Evident in Secondary Students' Concept Maps on Transformations of the Parabola

Winfilda Kapofu
University of Tasmania
winfilda.kapofu@utas.edu.au

Helen Chick
University of Tasmania
helen.chick@utas.edu.au

Vesife Hatisaru
Edith Cowan University
v.hatisaru@.ecu.edu.au

Carol Murphy
University of Tasmania
carol.murphy@utas.edu.au

This study examined the types of mathematical connections established by four secondary school students while constructing concept maps on transformations of the parabola. The use of concepts maps revealed students' understandings—including some misconceptions—of transformations of the parabola and confirmed the usefulness of a model for categorising types of connections.

For many, mathematics is a coherent and connected discipline characterised by a network of ideas. Some students, however, view it as a collection of separate entities (García-García & Dolores-Flores, 2018). Establishing connections between seemingly distinct mathematical ideas is central to conceptual understanding and is broadly recommended in research literature (Choy & Toh, 2021) for its ability, among other reasons, to integrate intra- and extra-mathematical knowledge (Rodríguez-Nieto et al., 2020). Also, we "understand something if we see how it is related or connected to other things we know" (Charles & Carmel, 2005, p. 10). For students to understand a new mathematical concept or acquire a new skill, they must connect their pre-existing understandings to the new concept or skills (Anthony & Walshaw, 2009). To assess students' understanding of a mathematical concept, an examination of how students connect the concept with other concepts can be undertaken (Barmby et al., 2009).

Research evidence (e.g., Yanik, 2014) shows that students struggle with the idea of transformations—including translations, reflections, rotations, and dilations—and hold ill-formed conceptual understandings of transformations (Hollebrands, 2004). Existing work on transformations of the parabola provides a basis for students' understanding of transformations of hyperbolic, exponential, and trigonometrical functions. In transformations of the parabola, many concepts can be represented both geometrically and algebraically, so it is important that students have the capacity to traverse between the two representations. The need to explore students' conceptual understandings of transformations of the parabola, therefore, has become critical. The research reported in this paper sought to answer the question: *What types of mathematical connections are evident in concept maps drawn by secondary school students among concepts associated with transformations of the parabola and other mathematical concepts?*

Mathematical Connections

There are two ways, among possible others, through which a deep understanding of a concept may be shown: (i) the connections a student makes between a concept and other mathematical ideas, and (ii) the student's various representations of the concept and the reasoning behind the connections made (Barmby et al., 2009). Borrowing from the definitions of mathematical connections available in the literature (Businskas, 2008; Eli et al., 2013; Garcia-Garcia & Dolores-Flores, 2018), in this paper, mathematical connections are considered to be the relationships a student constructs among mathematical ideas, representations, procedures, symbols, properties, definitions, and theorems. According to Businskas (2008), there are several ways of viewing a mathematical connection. These include: a relationship between ideas or processes; a process of making or recognizing links between

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mathematical ideas; associations between two or more mathematical ideas; and, finally, a causal or logical interdependence between mathematical entities. There has been extensive research on mathematical connections established by practicing (Businskas, 2008; Eli, Mohr-Schroeder, & Lee, 2013; Hatisaru, 2022; Mhlolo, 2012) and pre-service teachers (Evitts, 2004). Less attention has been given to mathematical connections established by students.

In her study, Businskas (2008) gave definitions for five types of mathematical connections identified by secondary teachers: different representations, implications, part-whole relationships, procedures, and instruction-oriented connections. *Different representations* are either alternative or equivalent representations of a concept. *Part-whole relationships* are such that one concept is contained in or is a component of the other. *Implications* are connections that highlight that A implies B. *Procedures* associate a process, or method, for working with a concept. *Instruction-oriented connections* indicate that A is a concept or skill necessary for understanding B, and these are associated particularly with teaching.

García-García and Dolores-Flores (2018) explored the mathematical connections established by high school students while solving calculus problems, using task-based interviews for data collection. García-García and Dolores-Flores' (2018) model consisted of seven types of mathematical connections, of which four were consistent with the model of Businskas (2008). As they worked with students, the instruction-oriented connections were not evident in their study. They added *feature connections* (also found earlier by Eli et al. (2011) and later by Hatisaru (2022)), evident when properties of a mathematical concept are presented or described in terms of what associates or differentiates it from other mathematical concepts, and *reversibility* and *meaning*. A *reversibility* connection is evident when a person is able to establish a two-way relationship between concepts. The *meaning* connection is manifested when the properties of a concept are linked to its rules, formulae, or processes including definitions and contexts.

In this study, an amalgamation of the model by Businskas (2008) and by García-García and Dolores-Flores (2018) was conjectured to be applicable for analysing mathematical connections identified by secondary school students. This model consists of connections that incorporate meaning, different representations, part-whole relationships, procedures, features, and reversibility. Further elaborations of these type of connections are presented in Table 1, which provides an outline of the types of connections related to the specific topic of transformations of the parabola.

Table 1

Types of Mathematical Connections Associated with Transformations of the Parabola

Type of connections	Examples
Meaning connections	Meanings or interpretations associated with concepts. Analogies for properties: when a familiar concept is linked to an abstract intended domain such as: (i) reflection over the x-axis as a “flip”; (ii) an upright parabolic shape as a “smile” or a “U shape”; or (iii) a translation of 5 units to the left as a “slide” or “shift” of 5 units
Different representations connections	Equivalent representations: (i) y-axis and $x = 0$ line; (ii) $y = ax^2 + bx + c$ and $y = a(x - h)^2 + k$ Alternate representations: e.g., A parabolic U shape in the Cartesian plane with key points labelled as an alternate representation of the function $y = x^2 - 3$
Part-whole relationship connections	For instance: (i) y- and x-intercepts and turning point are part of the parabola; (ii) a parabola has an axis of symmetry hence the axis of symmetry becomes part of the parabola

Type of connections	Examples
Procedure connections	For instance, $y = a(x - h)^2 + k$ a: gives the shape and gradient of the parabola: (i) reflection of the parabola over the x-axis (when a < 0); (ii) if |a| < 1 the parabola is wider and less steep than when a = 1; (iii) if |a| >1 the parabola is narrower and steeper than when a = 1 h: horizontal translation: to the right (when h > 0) or to the left (when h < 0), h $\in \mathbb{R}$ k: vertical translation: upward (when k > 0) or downward (when k < 0), k $\in \mathbb{R}$
Feature connections	Some or all the properties may be exhibited: The basic parabola has two roots, it is symmetrical about the y-axis and has a domain of x $\in \mathbb{R}$, (-∞, +∞) and a range of y $\in \mathbb{R}$, [0, +∞) or y ≥ 0
Reversibility connections	Forward and backward relationship. For instance: a parabola is given by $y = ax^2 + bx + c$ and the equation of the parabola can be obtained using information on the parabolic graph.

Methodology

This study is part of the first author's PhD investigation conducted in a year 10 'Mathematics General' class in an Australian secondary school. The mathematical content for this class enables above average students to continue on to a pre-tertiary mathematics course in year 11 and includes study of transformations of parabolas.

Concept maps are known to provide visual representations of dynamic structures of understanding within the human mind (Mls, 2004). A concept map is a visual technique reflecting the key perceptions of an individual regarding relationships between and among ideas (Wheeldon & Faubert, 2009). Concept maps were utilised as the data collection tool for this study in order to make visible each student's internally constructed connections associated with the transformations of the parabola.

Participants

Twelve sixteen-year-old students were the informants of the study. For the purpose of examining the types of connections associated with transformations of the parabola, the concept maps of four participants were purposefully selected for closer examination in this paper because three of the maps had additional elaborations and one, without additional elaborations, included many arrows indicating an extensive set of connections. The four participants were given pseudonyms: Enoch, Irvine, Nathan, and Jimmy. They had all been at the study school since year 7. Nathan and Jimmy were attending extension classes during Mathematics extended sessions. They intended pursuing a mathematics-oriented course at tertiary level. Enoch and Irvine did not intend to take up pre-tertiary mathematics courses.

Data Collection

Students received ten lessons of instruction on transformations of the parabola. Following this unit of work, participants were provided with concept cards for constructing their concept maps. The concept cards included concepts that were either related or unrelated to transformations of the parabola. There were also some blank cards. Participants were told to imagine they were writing a mathematics textbook on transformations of the parabola. They were given prompting questions to assist them with coming up with ideas they might wanted to include such as:

- How would you define the word "transformation" in mathematics?
- What are the different forms of transformation you know?
- Which topics are linked to transformations?
- What words, symbols, and representations are associated with transformations?

Participants could add their own concepts or information they felt had been omitted using the blank cards. Participants were encouraged to select and link their concepts and ideas with arrows and put labels on arrows to describe how the concepts were related. Some of the concept cards provided to the participants are seen in Figures 1 to 4.

Guided by the types of connections identified in Table 1, data were analysed to reveal the types of mathematical connections established by participants while they were constructing concept maps on transformations of the parabola.

Findings

All four students made connections among concepts in their concept maps. Participants used arrows or lines to connect related concepts and ideas; however, some of the participants provided clustered related concepts without using connecting arrows. In some instances, participants provided elaborations on the arrows to describe how they thought the ideas are related. Figure 1 illustrates how Enoch used arrows and elaborations to describe the connection between the parabola and each of the algebraic equations. In Figure 2, Nathan's concept map provides a fine example of clustered concepts. These were concepts he identified as connected by putting them in close proximity of each other without using arrows to connect them.

Meaning Connections

To some of the participants, "transformations of the parabola" meant *movement* of the parabola, changing its position on the Cartesian plane. This notion is captured, for instance, when Nathan puts "*position*" between "*flip*" and "*dilation*" and uses the phrases *"stretches along y or x axis"* to describe a dilation which is a form of transformation (see Figure 2). In the same concept map Nathan used certain phrases in his elaborations on horizontal shift: "$y = (x - h)^2$, *shifts h to the right"* and "$y = (x + h)^2$, *shifts h to the left"* (see Figure 2). Jimmy had a direct arrow from transformation to position then connected it with slide (see Figure 4), thus cementing the idea of transformation being perceived as movement. The meaning of translation as a form of transformation to all participants was either a vertical or a horizontal shift. This is evident in, for example, Irvine's concept map where he had arrows originating from translation to vertical shift and another to horizontal shift (see Figure 3). The meaning connection was also established as some participants clustered alternative words or phrases around a concept. For instance, *vertical shift* and *slide* were both linked to *translation* which is a form of transformation that can be described by these two terms (see Figure 2).

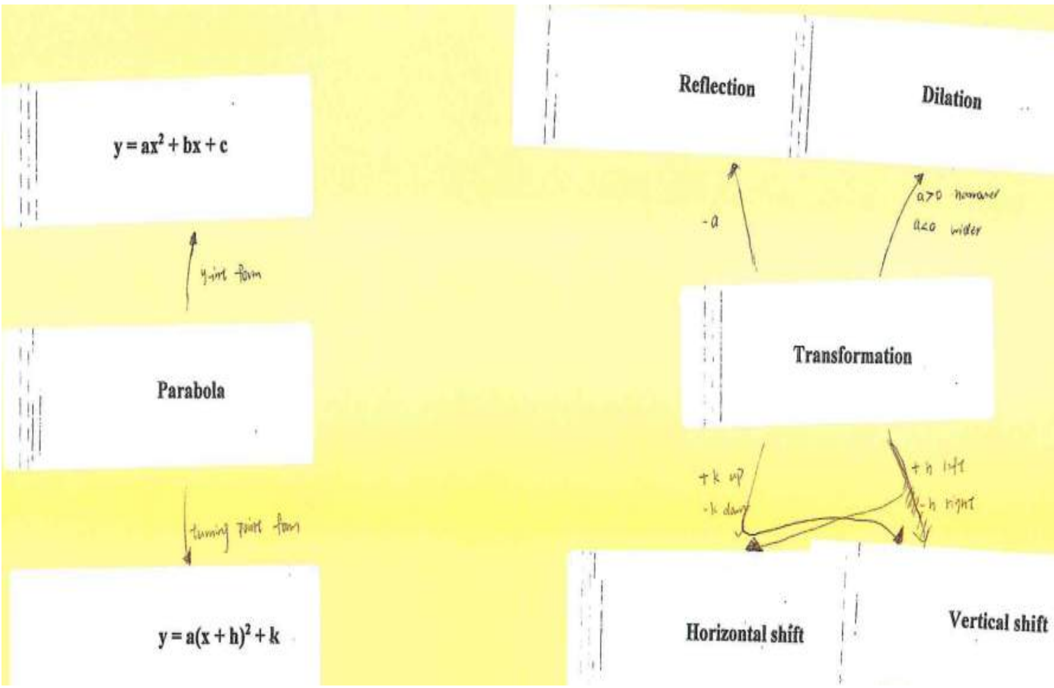

Figure 1. Enoch's concept map.

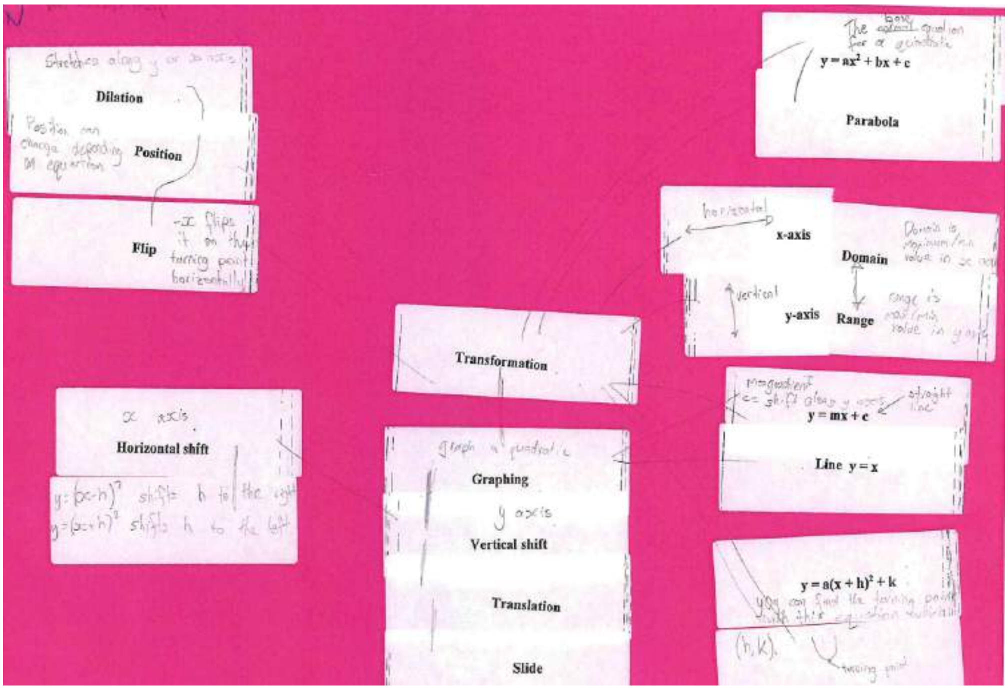

Figure 2. Nathan's concept map.

Different Representations Connections

Some participants established different representations connections (Businskas, 2008). In Figure 3, Irvine used a graphical representation to illustrate a translation of the quadratic function $y = x^2$, which was presented in the form of a solid line parabola and the resulting images after translation as a dotted line parabola for the vertical and the horizontal shifts. Enoch connected the equations $y = ax^2 + bx + c$ and $y = a(x + h)^2 + k$ as alternate representations of a parabola. He also used the parameter *-a* as an equivalent of a reflection, and *+k up and -k down* to represent a vertical shift (Figure 1). Nathan identified "$y = (x - h)^2$," as an alternate representation for the horizontal graphical translation of the parabola (Figure 2). In Figure 4, Jimmy identified $y = x$ and $y = mx + c$ as alternate representations of a line, although he also associated $y = ax^2 + bx + c$ as a line as well.

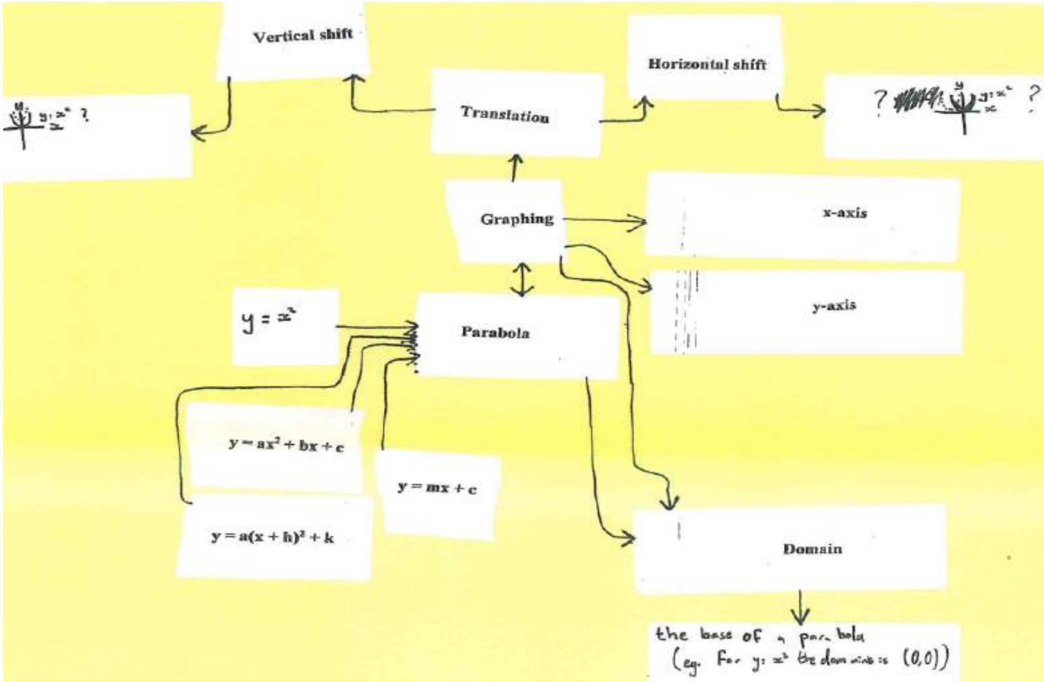

Figure 3. Irvine's concept map.

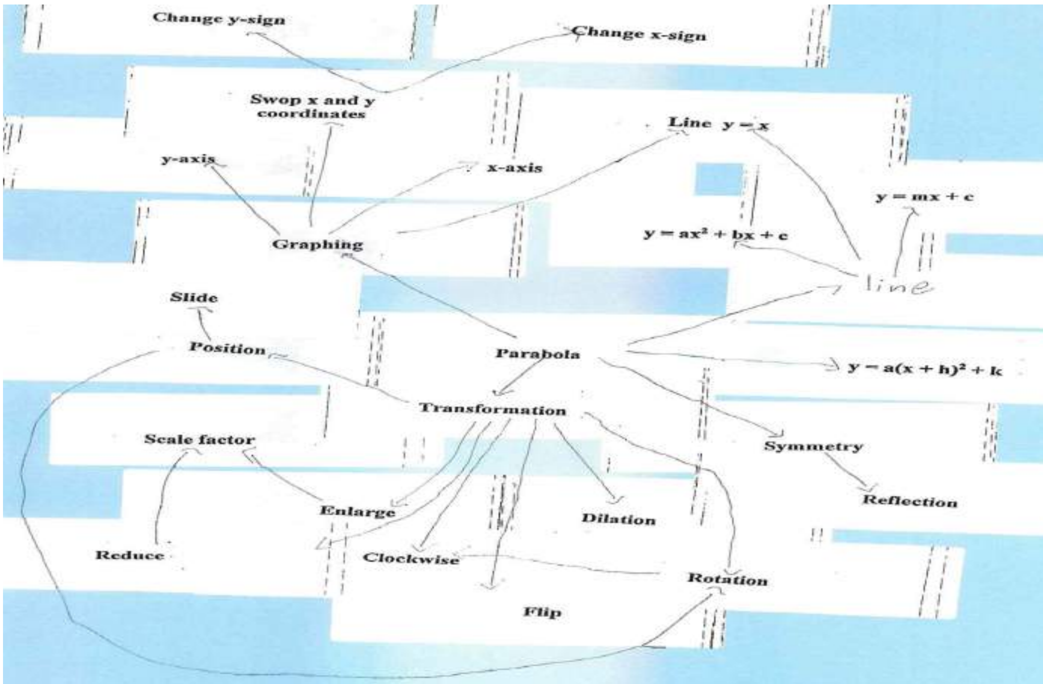

Figure 4. Jimmy's concept map.

Part-Whole Relationship Connections

Another connection type that surfaced was the part-whole relationship. This connection type was also found in Hatisaru (2022). In Figure 3, Irvine linked *x*- and *y-axes* to *graphing* making them part of the graphing process. In the same concept map, "parabola" was linked to "domain" meaning that it is part of the parabola. Jimmy connected "scale factor" to "enlarge" and "reduce" meaning that scale factor is a part of each one of them (Figure 4). Jimmy also connected "symmetry" to the parabola since the parabola is a symmetrical shape.

Procedure Connections

Procedural connections were evident in some participants' elaborations. For instance, Nathan gave the description of a flip as follows: *"-x flips it on the turning point horizontally"* (see Figure 2), suggesting that the shape (parabola) is reflected over the x-axis, implying that Nathan knew that the negative sign represents a reflection over the x-axis. Another emerging idea was that transformation influences the position of an object. This was evidenced by the connection made between *transformation* and *position* in Figure 2. This additional information: *"position can change depending on equation,"* showed that Nathan had developed an understanding that different parameters in an equation influence the outcome; thus, parameters in an equation determine processes required to be undertaken. Enoch connected transformation to dilation. He added the elaborations that "*a < 0 wider"* and *"a > 0 narrower"*, describing the process if one had to transform a shape by dilating it (see Figure 1, top right of concept map).

Feature Connections

Some of the participants associated transformations of the parabola with algebraic equations of quadratic functions. In Figure 2, Nathan associated the equation $y = ax^2 + bx + c$ with "parabola" by having the concept cards adjacent to each other and linked—with an arrow—"transformation" with the equation $y = a(x + h)^2 + k$, where he highlighted the role of h and k in determining the turning point, and thus the translation of the basic parabola $y = x^2$. He also connected "transformation" to the $y = ax^2 + bx + c$ equation, although it was not clear how he thought transformation is associated with that equation. Irvine made connections between the parabola and its various forms of representation and the general form of a straight line (see Figure 3). These connections reveal that, besides being able to identify and describe the transformation, Irvine had the ability to navigate between the algebraic and geometric domains utilising the feature connection.

Reversibility Connections

The reversibility connection from García-García and Dolores-Flores (2018) was only evident in one concept map and on one occasion. Bi-directional arrows were drawn to connect transformation to some of its forms. For example: *graphing ↔ parabola as* seen in Irvine's map in Figure 3 indicates that the relationship between graphing and parabola is reversible. This suggests that Irvine understood that one could use given information to graph a parabola. At the same time, a person could extract information from the graphical form of a parabola.

Discussion and Conclusions

The types of connections established by participating students were consistent with those from the models of Businskas (2008) and García-García and Dolores-Flores (2018), with the exceptions of the instruction-oriented and implication connections. This may be attributed to the differences in the sample type as well as the data collection techniques. It was not expected to identify the instruction-oriented connection in the students' works. Implication connections and procedural connections were difficult to identify in the absence of detailed elaborations.

The students exhibited types of transformations using alternative wording and flexibility in the interpretation of algebraic transformation equations. Nevertheless, there were gaps in connecting concepts to their formal definitions. For example, the connection between "flip" and its formal name (reflection) was not evident in students' work. Further elaborations revealed some misconceptions held by the students. This was perhaps best evident in Nathan's response stating that "range is max/min value in the y axis" (Figure 2). It was also observed that participants made some ambiguous connections in their concept maps. For instance, in Figure 4, Jimmy did not provide descriptions of how the linked concepts were related. In other words, the concept map did not include enough information for some of the connections he might have established. Clearly much detailed information—such as from interviews—might be needed to gain better insights into the connections that students make.

These findings indicate that, as a data collecting tool, the concept map does not capture all the types of mathematical connections established by a student to reflect the full extent of their understanding of a learnt concept. Considering the emerging issues in this study, it is proposed that for most of the established connections to be captured, the concept maps could be accompanied by elaborations, and interviews, which would provide the participant with an opportunity to reveal some of the missing connections through their explanations and descriptions.

With the emerging issues mentioned above in mind, findings from this study can provide teachers with insight into what students regard as connections between and among concepts, and plausibly can assist in identifying misconceptions and gaps in students' conceptual understandings. They can also trigger teachers to reflect on their personal understandings of concepts creating opportunities for helping students develop well-formed conceptual understandings. Finally, education practitioners will be enlightened on how the formal concept definitions may be interpreted by students in personal ways.

References

Businskas, A. M. (2008). *Conversations about connections: How secondary mathematics teachers conceptualize and contend with mathematical connections* [Master's thesis, Simon Fraser University].

Charles, R. I., & Carmel, C. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education, 7*(3), 9–24.

Eli, J. A., Mohr-Schroeder, M. J., & Lee, C. W. (2013). Mathematical connections and their relationship to mathematics knowledge for teaching geometry. *School Science and Mathematics, 113*(3), 120–134.

Evitts, T. A. (2004). *Investigating the mathematical connections that preservice teachers use and develop while solving problems from reform curricula* [Doctoral thesis, The Pennsylvania State University].

García-García, J., & Dolores-Flores, C. (2018). Intra-mathematical connections made by high school students in performing Calculus tasks. *International Journal of Mathematical Education in Science and Technology, 49*(2), 227–252.

Hatisaru, V. (2022). Mathematical connections established in the teaching of functions. *Teaching Mathematics and its Applications*: *An International Journal of the IMA*1-21. https://doi.org/10.1093/teamat/hrac013

Hollebrands, K. F. (2004). Connecting research to teaching: High school students' intuitive understandings of geometric transformations. *The Mathematics Teacher, 97*(3), 207–214.

Mhlolo, M. K. (2012). Mathematical connections of a higher cognitive level: A tool we may use to identify these in practice. *African Journal of Research in Mathematics, Science and Technology Education, 16*(2), 176–191.

Mls, K. (2004). From concept mapping to qualitative modeling in cognitive research. *In concept maps: Theory, methodology, technology: Proceedings of the first international conference on concept mapping* (pp. 271–274).

Rodríguez-Nieto, C. A., Rodríguez-Vásquez, F. M., & Moll, V. F. (2020). A new view about connections: The mathematical connections established by a teacher when teaching the derivative. *International Journal of Mathematical Education in Science and Technology*, 53, 1231–1256. https://doi.org/10.1080/0020739X.2020.1799254

Toh, T. L., & Choy, B. H. (Eds.) (2021). *Mathematics connection and beyond: Yearbook (2020)*. Association of Mathematics Educators. World Scientific.

Yanik, H. B. (2014). Middle-school students' concept images of geometric translations. *The Journal of Mathematical Behavior, 36,* 33–50.